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Inadequacy of First-Order Upwind Difference Schemes for Some Recirculating Flows*

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Spurious numerical solutions of problems with closed sub-characteristics by upwind difference schemes, in particular problems of recirculating incompressible flow at high Reynolds numbers, are proved to be due to the anisotropy of the artificial viscosity. Numerical examples are presented to show that even very simple problems, including basic problems in fluid dynamics, are not approximated well by schemes with anisotropic artificial viscosity, regardless of numerical parameters. © 1991 Academic Press, Inc.

1. INTRODUCTION

Numerical methods for solving incompressible fluid flow equations are frequently tested on problems of recirculating flow. Quite often results obtained by different methods differ from each other significantly. Reliable experimental data is usually difficult to obtain, and discrepancies between numerical two-dimensional solutions and experimental data may be the result of the influence of walls in the experiments.

The problems in solving recirculating incompressible flows at high Reynolds numbers are well known. Several researchers have pointed out that some of the methods employed to obtain stable discretization, e.g., upwind differencing, may lead to spurious results. Such results have been attributed to excessive artificial viscosity in the numerical scheme, to multiple solutions of the non-linear set of algebraic equations that is obtained from the numerical scheme, and to poor resolution by grids that are too coarse [2, 4, 5]. But spurious results occur even when none of these reasons are valid. Indeed, a single linear partial differential equation may exhibit such behavior when the sub-characteristics are closed. This is true even when the mesh size is small enough to easily resolve the solution and truncation errors are genuinely small everywhere, and even when the differential solution does not depend on the *size* of the viscosity coefficient at all.

It is well known that the problem of shear driven recirculating flow is not well

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posed when the viscous terms vanish. The system then loses its ellipticity, and the boundary conditions are no longer appropriate [1]. Thus, the viscous terms may play an important role in determining the solution throughout the domain, even when their absolute values tend to zero. In effect, when the coefficient of viscosity is small (high Reynolds numbers), the convection terms dictate the behavior of the solution *along* the streamlines, while the viscosity determines its variation *across* streamlines. Since the boundary itself is a streamline, the propagation of information from the boundary into the domain is governed by the viscous terms, no matter how small their coefficients may be. Hence, the manner in which these coefficients tend to zero may effect the solution significantly. Anisotropic artificial viscosity, i.e., different coefficients for the second derivative terms may produce results that differ considerably from isotropic viscosity solutions to the *differential* problem. Obviously, this is equally true for linear problems and is independent of numerical parameters such as coarseness of the grid.

It should be noted that the phenomenon presented here is only possible in two or more dimensions. Since there is no one-dimensional analogue of recirculation, the particular ill-posedness described does not occur in one dimension. Also, the problems do not appear when the grid is consistently aligned with the characteristic directions, and the concept of non-alignment is not present in one-dimensional problems.

2. Spurious Upwind Solution for a Linear PDE

In order to give an indication of difficulties that arise even in relatively simple cases of problems with closed sub-characteristics, we consider the following equation and boundary conditions, written in polar coordinates (r, θ):

$$\begin{aligned} -\varepsilon\,\Delta U + \frac{1}{r^2}\frac{\partial U}{\partial \theta} &= 0 \qquad (r, \theta) \in \Omega \\ U(r=a, \theta) &= U_i \\ U(r=b, \theta) &= U_0, \end{aligned} \tag{2.1}$$

where

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

is the Laplacian, ε a positive constant, and Ω is disk of radius b with a circular hole of radius a in its center. U_i and U_0 are given constants.

The unique solution of (2.1), easily obtained by separation of variables, is

$$U = \ln\left[\left(\frac{r}{a}\right)^{U_0}\left(\frac{b}{r}\right)^{U_i}\right] \Big/ \ln\left(\frac{b}{a}\right). \tag{2.2}$$

It is seen that the solution is constant over the circular sub-characteristics. While it is independent of ε (in the special case of constant boundary values), its *cross-characteristic* behavior is nonetheless determined by the diffusive terms.

In order to see what might go wrong with the numerical solution of this problem, we rewrite the equation in cartesian coordinates (x, y),

$$-\varepsilon \Delta U - \frac{y}{x^2+y^2}\frac{\partial U}{\partial x} + \frac{x}{x^2+y^2}\frac{\partial U}{\partial y} = 0, \tag{2.3}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

As is well known, discretization of this equation by central finite differences loses its stability when ε is small compared to the product of the mesh size and the absolute value of either coefficient of the first derivatives in the equation. A common practice is to retain stability by increasing the absolute values of the coefficients of the discretized second derivatives. A particular method of this type is first-order upwind differencing. In effect, this sort of discretization gives a second-order approximation to the equation

$$-\varepsilon_1 \frac{\partial^2 \tilde{U}}{\partial x^2} - \varepsilon_2 \frac{\partial^2 \tilde{U}}{\partial y^2} - \frac{y}{x^2+y^2}\frac{\partial \tilde{U}}{\partial x} + \frac{x}{x^2+y^2}\frac{\partial \tilde{U}}{\partial y} = 0, \tag{2.4}$$

where ε_1 and ε_2 are functions of x and y. Transforming back to polar coordinates, we obtain

$$\begin{aligned} &-(\varepsilon_1 \cos^2\theta + \varepsilon_2 \sin^2\theta)\frac{\partial^2 \tilde{U}}{\partial r^2} - (\varepsilon_1 \sin^2\theta + \varepsilon_2 \cos^2\theta)\frac{1}{r}\left(\frac{\partial \tilde{U}}{\partial r} + \frac{1}{r}\frac{\partial^2 \tilde{U}}{\partial \theta^2}\right) \\ &\quad - \sin 2\theta(\varepsilon_2 - \varepsilon_1)\frac{1}{r}\left(\frac{\partial^2 \tilde{U}}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial \tilde{U}}{\partial \theta}\right) + \frac{1}{r^2}\frac{\partial \tilde{U}}{\partial \theta} = 0, \end{aligned} \tag{2.5}$$

where, in the particular case of first-order upwing differencing, $\varepsilon_1 = \varepsilon + (h_x/2r)|\sin\theta|$ and $\varepsilon_2 = \varepsilon + (h_y/2r)|\cos\theta|$, h_x and h_y being the mesh sizes in the x and y directions, respectively. We choose for simplicity $h_x = h_y = h$. Note that problems of instability occur only when ε is small compared to h. Thus, we assume $\varepsilon \leqslant h/2$, since otherwise artificial viscosity is unnecessary. The exact bound on ε with respect to h is not crucial in the proof below, but it is important that h be at least $O(\varepsilon)$. If the mesh size is much smaller than ε, the problems described will not occur. But such mesh sizes are very much smaller than is necessary to obtain good resolution and therefore yield extremely inefficient solvers. Hence, the range of mesh sizes examined is precisely that for which upwind differencing is useful.

Let us now define the function V to be the difference between the solutions of (2.5) and (2.1). From (2.2) we obtain the following differential equation for V, after substituting the expressions for ε_1 and ε_2 and multiplying by $2r/h$,

$$-\left(f(\theta)+\frac{2\varepsilon r}{h}\right)\frac{\partial^2 V}{\partial r^2}-\left(g(\theta)+\frac{2\varepsilon r}{h}\right)\left(\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}\right)$$

$$-k(\theta)\left(\frac{1}{r}\frac{\partial^2 V}{\partial r\,\partial \theta}-\frac{1}{r^2}\frac{\partial V}{\partial \theta}\right)+\frac{2}{hr}+\frac{\partial V}{\partial \theta}=m(\theta)\frac{C}{r^2}. \tag{2.6}$$

$$V(r=a,\theta)=V(r=b,\theta)=0,$$

where

$$f(\theta)=|\sin\theta\cos^2\theta|+|\cos\theta\sin^2\theta|$$

$$g(\theta)=|\cos^3\theta|+|\sin^3\theta|$$

$$k(\theta)=\sin 2\theta\cdot(|\cos\theta|-|\sin\theta|)$$

$$m(\theta)=g(\theta)-f(\theta)=(|\cos\theta|+|\sin\theta|)\cdot(|\cos\theta|-|\sin\theta|)^2\geqslant 1-|\sin 2\theta|\geqslant 0$$

$$C=\frac{U_0-U_i}{\ln(b/a)}.$$

Let us now consider a problem where the order of magnitude of U_0-U_i, a, and $b-a$ is 1 (denoted $O(1)$) as $h\to 0$. A similar, albeit more lengthy calculation can be made for greater mesh sizes too, but it is not of practical interest.

We now show that V is of substantial size throughout Ω except near the boundaries, where it vanishes, of course. This implies that the upwind difference scheme cannot approximate (2.1) well, regardless of numerical parameters. The proof relies on the maximum principle, which enables us to bound V from below by a polynomial in the r variable, which is positive everywhere in the interior of Ω.

CLAIM. *For $h\to 0$ and $\varepsilon\leqslant h/2$, $V(r,\theta)\leqslant\mu\bar r^3(1-\bar r)^3-O(h)$ throughout Ω, where μ is an $O(1)$ constant calculated below, and $\bar r=(r-a)/(b-a)$ is a normalized r variable.*

Proof. The linear operator at the left-hand side of (2.6) is elliptic. We shall denote it by $\mathscr{L}$. Let us define

$$W(r,\theta)=\mu\bar r^3(1-\bar r)^3+\frac{hC}{8b}\cdot[\sin 2\pi\bar r+\bar r(1-\bar r)\sin 4\theta]. \tag{2.7}$$

Our object is to choose μ so that

$$\mathscr{L}(V(r,\theta)-W(r,\theta))\geqslant 0 \qquad (r,\theta)\in\Omega. \tag{2.8}$$

Due to the periodicity of m, W, and $\mathscr{L}$ we may restrict our calculations to $0\leqslant\theta\leqslant\pi/2$. By (2.6) it suffices to prove

$$\mathscr{L}(W(r,\theta))\leqslant\frac{C}{r^2}(1-\sin 2\theta);$$

i.e.,

$$\{-6F(r,\theta)\cdot[(1-\bar{r})^2-3\bar{r}(1-\bar{r})+\bar{r}^2] \\ -3G(r,\theta)\cdot\bar{r}(1-\bar{r})(1-2\bar{r})\}\cdot\bar{r}(1-\bar{r})\mu \\ \leqslant\frac{C}{r^2}(1-\sin 2\theta)-\frac{C}{br}\bar{r}(1-\bar{r})\cos 4\theta-h\phi(r,\theta), \tag{2.9}$$

where

$$F(r,\theta)=\frac{f(\theta)+2\varepsilon r/h}{(b-a)^2},\qquad G(r,\theta)=\frac{g(\theta)+2\varepsilon r/h}{(b-a)r}$$

and

$$\phi(r,\theta)=\frac{C}{8b}\cdot\left\{2F(r,\theta)+G(r,\theta)\cdot\left[2\bar{r}-1+16\bar{r}(1-\bar{r})\frac{b-a}{r}\right]\right\}\sin 4\theta \\ +\frac{C}{2b}\cdot\frac{k(\theta)}{r}\cdot\left[\frac{2\bar{r}-1}{b-a}+\frac{1}{r}\bar{r}(1-\bar{r})\right]\cos 4\theta \\ +\frac{C}{8b}\cdot[4\pi^2F(r,\theta)\sin 2\pi\bar{r}-2\pi G(r,\theta)\cos 2\pi\bar{r}].$$

Consider the expression

$$\frac{h\phi(r,\theta)}{(C/r^2)(1-\sin 2\theta)-(C/br)\,\bar{r}(1-\bar{r})\cos 4\theta}$$

in the interior of the domain. The denominator is positive everywhere. Furthermore, it is $O(1)$ everywhere in the first quadrant except where $\theta\to\pi/4$ and $\bar{r}(1-\bar{r})\to 0$, in which case ϕ is negative. Therefore the right-hand side of (2.9) may be replaced by

$$\left[\frac{C}{r^2}(1-\sin 2\theta)\frac{C}{br}\bar{r}(1-\bar{r})\cos 4\theta\right]\cdot(1-O(h)).$$

Equating the θ derivative of this expression to zero, we obtain

$$\left[-\frac{C}{r}+\frac{4C}{b}\bar{r}(1-\bar{r})\sin 2\theta\right]\cdot\cos 2\theta=0.$$

For every $a<r<b$ the minimum of the expression is obtained when $\theta=\pi/4$. It is enough, therefore, to choose μ that will satisfy

$$\{-6F(r,\theta)\cdot[(1-\bar{r})^2-3\bar{r}(1-\bar{r})+\bar{r}^2]-3G(r,\theta)\cdot\bar{r}(1-\bar{r})(1-2\bar{r})\} \\ \cdot\mu\leqslant\frac{C}{br}-O(h).$$

The left-hand side terms can be estimated

$$-6F(r,\theta)\cdot[(1-\bar{r})^2-3\bar{r}(1-\bar{r})+\bar{r}^2]\leqslant -6\cdot\frac{\sqrt{0.5}+b}{(b-a)^2}\cdot\left(-\frac{1}{4}\right)$$

and

$$-3G(r,\theta)\cdot\bar{r}(1-\bar{r})(1-2\bar{r})\leqslant -3\cdot\frac{1+b}{a(b-a)}\cdot\left(-\frac{\sqrt{3}}{18}\right)$$

from which we obtain that

$$\mu\leqslant C\Big/b^2\cdot\left[\frac{3}{2}\frac{\sqrt{0.5}+b}{(b-a)^2}+\frac{\sqrt{3}}{6}\frac{1+b}{a(b-a)}\right]-O(h)$$

is sufficient to satisfy (2.8). Since $V-W=0$ on the boundaries of Ω, and, by the maximum principle, cannot have a minimum in the interior, V must be greater than W throughout Ω.

Thus it is proved that the upwind difference solution is totally inadequate, yielding an $O(1)$ error everywhere except near the boundaries. In fact, since the maximum principle is generalizable to difference schemes, this proof may be applied directly to the difference equations.

3. Incompressible Navier–Stokes

It seems likely that the problems that occur in the case of a linear equation will be present, and even more so, in a coupled non-linear system. However, due to the greater complexity of the system and its solutions, these problems might be harder to identify and much harder to analyze. Still, even the simple and well-known problem of steady incompressible two-dimensional flow at high Reynolds numbers between concentric rotating cylinders can be shown *not* to yield a first-order accurate solution when solved with upwind differencing on a cartesian grid. The steady incompressible Navier–Stokes equations in cartesian coordinates are

$$-\varepsilon\,\Delta U_x+U_x\frac{\partial U_x}{\partial x}+U_y\frac{\partial U_x}{\partial y}+\frac{\partial P}{\partial x}=0 \tag{3.1a}$$

$$-\varepsilon\,\Delta U_y+U_x\frac{\partial U_y}{\partial x}+U_y\frac{\partial U_y}{\partial y}+\frac{\partial P}{\partial y}=0 \tag{3.1b}$$

$$\frac{\partial U_x}{\partial x}+\frac{\partial U_y}{\partial y}=0, \tag{3.1c}$$

where U_x and U_y are the velocities in the x and y directions, respectively, P is the pressure variable, and $\varepsilon=1/R$, R being the Reynolds number. Let us rewrite these equations with anisotropic viscosity coefficients,

$$-\varepsilon_1 \frac{\partial^2 \tilde{U}_x}{\partial x^2} - \varepsilon_2 \frac{\partial^2 \tilde{U}_x}{\partial y^2} + \tilde{U}_x \frac{\partial \tilde{U}_x}{\partial x} + \tilde{U}_y \frac{\partial \tilde{U}_x}{\partial y} + \frac{\partial \tilde{P}}{\partial x} = 0 \qquad (3.2a)$$

$$-\varepsilon_1 \frac{\partial^2 \tilde{U}_y}{\partial x^2} - \varepsilon_2 \frac{\partial^2 \tilde{U}_y}{\partial y^2} + \tilde{U}_x \frac{\partial \tilde{U}_y}{\partial x} + \tilde{U}_y \frac{\partial \tilde{U}_y}{\partial y} + \frac{\partial \tilde{P}}{\partial y} = 0 \qquad (3.2b)$$

$$\frac{\partial \tilde{U}_x}{\partial x} + \frac{\partial \tilde{U}_y}{\partial y} = 0, \qquad (3.2c)$$

where ε_1 and ε_2 are functions of x and y.

Transformations of system (3.1) to polar coordinates yields

$$-\varepsilon\left(\Delta U_r - \frac{U_r}{r^2} - \frac{2}{r^2}\frac{\partial U_\theta}{\partial \theta}\right) + U_r \frac{\partial U_r}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_r}{\partial \theta} + \frac{\partial P}{\partial r} - \frac{U_\theta^2}{r} = 0 \qquad (3.3a)$$

$$-\varepsilon\left(\Delta U_\theta - \frac{U_\theta}{r^2} + \frac{2}{r^2}\frac{\partial U_r}{\partial \theta}\right) + U_r \frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial P}{\partial \theta} + \frac{U_r U_\theta}{r} = 0 \qquad (3.3b)$$

$$\frac{\partial (r U_r)}{\partial r} + \frac{\partial U_\theta}{\partial \theta} = 0, \qquad (3.3c)$$

U_r and U_θ being the velocities in the r and θ directions. Similarly, transformation of system (3.2) yields

$$\begin{aligned}
&-(\varepsilon_1 \cos^2\theta + \varepsilon_2 \sin^2\theta)\frac{\partial^2 \tilde{U}_r}{\partial r^2} \\
&-(\varepsilon_2 \cos^2\theta + \varepsilon_1 \sin^2\theta)\left[\frac{1}{r}\frac{\partial \tilde{U}_r}{\partial r} + \frac{1}{r^2}\left(-\tilde{U}_r + \frac{\partial^2 \tilde{U}_r}{\partial \theta^2} - 2\frac{\partial \tilde{U}_\theta}{\partial \theta}\right)\right] \\
&-\sin 2\theta(\varepsilon_2 - \varepsilon_1)\frac{1}{r}\left[\frac{\partial^2 \tilde{U}_r}{\partial r\,\partial \theta} - \frac{\partial \tilde{U}_\theta}{\partial r} - \frac{1}{r}\left(\frac{\partial \tilde{U}_r}{\partial \theta} - \tilde{U}_\theta\right)\right] \\
&+\tilde{U}_r \frac{\partial \tilde{U}_r}{\partial r} + \frac{\tilde{U}_\theta}{r}\frac{\partial \tilde{U}_r}{\partial \theta} + \frac{\partial \tilde{P}}{\partial r} - \frac{\tilde{U}_\theta^2}{r} = 0 \qquad (3.4a)
\end{aligned}$$

$$\begin{aligned}
&-(\varepsilon_1 \cos^2\theta + \varepsilon_2 \sin^2\theta)\frac{\partial^2 \tilde{U}_\theta}{\partial r^2} \\
&-(\varepsilon_2 \cos^2\theta + \varepsilon_1 \sin^2\theta)\left[\frac{1}{r}\frac{\partial \tilde{U}_\theta}{\partial r} + \frac{1}{r^2}\left(-\tilde{U}_\theta + \frac{\partial^2 \tilde{U}_\theta}{\partial \theta^2} + 2\frac{\partial \tilde{U}_r}{\partial \theta}\right)\right] \\
&-\sin 2\theta(\varepsilon_2 - \varepsilon_1)\frac{1}{r}\left[\frac{\partial^2 \tilde{U}_\theta}{\partial r\,\partial \theta} + \frac{\partial \tilde{U}_r}{\partial r} - \frac{1}{r}\left(\frac{\partial \tilde{U}_\theta}{\partial \theta} + \tilde{U}_r\right)\right] \\
&+\tilde{U}_r \frac{\partial \tilde{U}_\theta}{\partial r} + \frac{\tilde{U}_\theta}{r}\frac{\partial \tilde{U}_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial \tilde{P}}{\partial \theta} + \frac{\tilde{U}_\theta \tilde{U}_r}{r} = 0 \qquad (3.4b)
\end{aligned}$$

$$\frac{\partial (r\tilde{U}r)}{\partial r} + \frac{\partial \tilde{U}_\theta}{\partial \theta} = 0. \qquad (3.4c)$$

Let us consider the problem of flow between rotating cylinders of radii a and b, where $0<a<b$. Let us choose the boundary conditions so as to give a solution of zero in vorticity to system (3.3), i.e.,

$$U_r(r=a,\theta)=0, \qquad U_\theta(r=a,\theta)=V/a,$$
$$U_r(r=b,\theta)=0, \qquad U_\theta(r=b,\theta)=V/b,$$

V being a constant, System (3.3) with the above boundary conditions has the exact solution:

$$U_\theta=\frac{V}{r}, \qquad U_r=0, \qquad P=-\frac{V^2}{2r^2}+P_0. \tag{3.5}$$

The numerical solution of this problem with an upwind difference scheme on a cartesian grid involves a second-order accurate approximation of system (3.4) with $\varepsilon_1=\varepsilon+\frac{1}{2}|U_x|h_x$ and $\varepsilon_2=\varepsilon+\frac{1}{2}|U_y|h_y$, where h_x and h_y are the mesh sizes in the x and y directions, respectively. For simplicity we shall assume below that $h_x=h_y=h$.

If the solution of the numerical system of equations is to yield a first-order approximation to the differential problem, then the difference between the differential solutions of systems (3.3) and (3.4) must clearly tend to zero with h, and at the same rate at least. Let us define accordingly,

$$u_\theta=\tilde{U}_\theta-\frac{V}{r}, \qquad u_r=\tilde{U}_r, \qquad p=\tilde{P}+\frac{V^2}{2r^2}-P_0, \tag{3.6}$$

and assume that u_r, u_θ, and p tend to zero with h. Thus, we may neglect the terms involving products of these variables and h in the calculation (since they can only add to (3.10) terms that are at most of the same order of magnitude as its first term) and assume

$$\varepsilon_1=\frac{Vh|\sin\theta|}{2r}+\varepsilon, \qquad \varepsilon_2=\frac{Vh|\cos\theta|}{2r}+\varepsilon.$$

Substitution into (3.4b) yields

$$\begin{aligned}
&-\left(\frac{Vhf(\theta)}{2r}+\varepsilon\right)\left(\frac{2V}{r^3}+\frac{\partial^2 u_\theta}{\partial r^2}\right)\\
&-\left(\frac{Vhg(\theta)}{2r}+\varepsilon\right)\left[-\frac{V}{r^3}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}+\frac{1}{r^2}\left(-\frac{V}{r}-u_\theta+\frac{\partial^2 u_\theta}{\partial\theta^2}+2\frac{\partial u_r}{\partial\theta}\right)\right]\\
&-\frac{Vhk(\theta)}{2r}\frac{1}{r}\left[\frac{\partial^2 u_\theta}{\partial r\,\partial\theta}+\frac{\partial u_r}{\partial r}-\frac{1}{r}\left(\frac{\partial u_\theta}{\partial\theta}+u_r\right)\right]\\
&+u_r\left(-\frac{V}{r^2}+\frac{\partial u_\theta}{\partial r}\right)+\left(\frac{V}{r^2}+\frac{u_\theta}{r}\right)\frac{\partial u_\theta}{\partial\theta}+\frac{1}{r}\frac{\partial p}{\partial\theta}+\left(\frac{V}{r}+u_\theta\right)\frac{u_r}{r}=0,
\end{aligned} \tag{3.7}$$

$f(\theta)$, $g(\theta)$, and $k(\theta)$ defined as in (2.6).

Integration of (3.7) along circles of radii r, $a \leqslant r \leqslant b$, yields

$$-\frac{Vh}{2}\int_0^{2\pi}\left[f(\theta)\frac{\partial^2 u_\theta}{\partial r^2}+\frac{g(\theta)}{r}\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)+\frac{g''(\theta)}{r^2}u_\theta-2\frac{g'(\theta)}{r^2}u_r\right]d\theta$$
$$-\varepsilon r\int_0^{2\pi}\left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}\right)d\theta+\int_0^{2\pi}\left(ru_r\frac{\partial u_\theta}{\partial r}+u_r u_\theta\right)d\theta=-\frac{8}{3}\frac{V^2 h}{r^3}, \qquad (3.8)$$

where $g'(\theta)$ and $g''(\theta)$, the first and second derivatives of $g(\theta)$, result from integration by parts. Note that they are bounded throughout Ω. From the symmetry of the equations, and indeed the problem itself, the solutions u, v, and p are periodic over an interval of $\pi/2$, which accounts for the vanishing of the term multiplied by $k(\theta)$ in (3.7). Defining

$$\gamma(r)=\int_0^{2\pi} u_\theta\,d\theta$$

$$\varphi(r)=\int_0^{2\pi} f(\theta)\,u_\theta\,d\theta$$

$$\psi(r)=\int_0^{2\pi} g(\theta)\,u_\theta\,d\theta$$

$$\kappa(r)=\int_0^{2\pi} g'(\theta)\,u_r\,d\theta$$

$$\eta(r)=\int_0^{2\pi} g''(\theta)\,u_\theta\,d\theta,$$

multiplying (3.8) by r and exchanging the orders of integration and differentiation we obtain

$$-\frac{Vh}{2}\cdot\left[(r\varphi'(r))'+\psi'(r)-\varphi'(r)-\frac{\psi(r)}{r}+\frac{\eta(r)}{r}-2\frac{\kappa(r)}{r}\right]$$
$$-\varepsilon r\cdot\left[(r\gamma'(r))'-\frac{\gamma(r)}{r}\right]+\int_0^{2\pi}\left(r^2u_r\frac{\partial u_\theta}{\partial r}+ru_r u_\theta\right)d\theta=-\frac{8}{3}\frac{V^2h}{r^2}. \qquad (3.9)$$

Let us define

$$\|\cdot\|=\int_a^b\int_0^{2\pi}|\cdot|r\,d\theta\,dr.$$

We now integrate (3.9) thrice (for explicit calculation see Appendix A), and estimate the various terms. For ε that is not large compared to $V\cdot h$, we obtain the following equation in orders of magnitude:

$$h \cdot O(\|u_r\| + \|u_\theta\|) + O(\|u_r\| \cdot \|u_\theta\|)$$
$$= h \cdot \left[K(R-a)^2 - \frac{8V^2}{3} \left(\frac{R^2 - a^2}{2a} - R \ln \frac{R}{a} \right) \right]. \tag{3.10}$$

Here K is a constant that results from integration by parts, and the equation holds for any value of R between a and b.

Since the right-hand side of (3.10) varies by $O(h)$ for different values of R, it is clear that the L_1 norms of u_r and u_θ cannot be $O(h)$ as well. In fact one of them at least must be $O(\sqrt{h})$ or larger. Actually, there is nothing to suggest that u_r and u_θ converge to zero at all.

4. Numerical Examples

4.1. *Linear Equation*

Since problems with circular sub-characteristics will normally be solved in polar coordinates, in which case the difficulty described above does not appear due to consistent alignment, a more natural domain Ω for cartesian representation was chosen in the numerical example: a square of side 1 centered at the origin with a hole in its middle, such that both the inner and the outer boundaries are sub-characteristics. The partial differential equation and boundary conditions

$$-\varepsilon \Delta U + F(x, y) \frac{\partial U}{\partial x} + G(x, y) \frac{\partial U}{\partial y} = 0 \qquad (x, y) \in \Omega \tag{4.1}$$
$$U|_{\partial_i \Omega} = 0$$
$$U|_{\partial_0 \Omega} = 1$$

were solved, where

$$\partial_i \Omega = \{(x, y)\colon \cos \pi x \cos \pi y = \sqrt{0.5}, |x|, |y| \leqslant \tfrac{1}{2}\}$$

and

$$\partial_0 \Omega = \{(x, y)\colon \cos \pi x \cos \pi y = 0, |x|, |y| \leqslant \tfrac{1}{2}\}$$

are the inner and outer boundaries of Ω, respectively. $F(x, y) = \sin \pi y \cos \pi x$ and $G(x, y) = -\cos \pi y \sin \pi x$ were chosen so as to give contours of $\cos \pi x \cos \pi y$ as the closed sub-characteristics (Fig. 1).

The equations were discretized on a uniform cartesian grid of mesh size h in both x and y directions, except at the inner boundary, where grid points were defined on the boundary. Several values of h were tried, the finest being $\frac{1}{128}$. The second-order five point star scheme was used for the Laplacian, and the discretization was modified near the inner boundary, where appropriate, in accordance with the

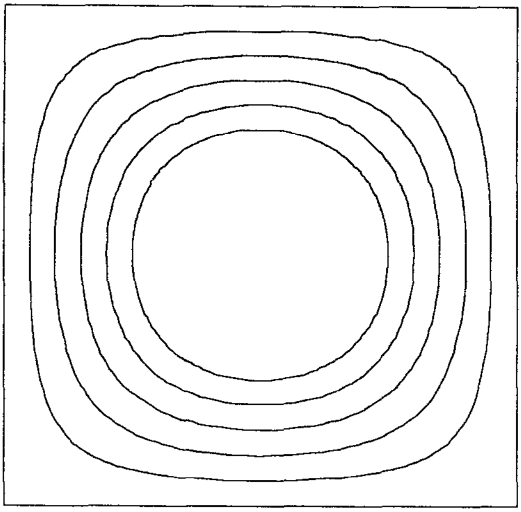

FIG. 1. Sub-characteristics of (4.1).

reduced mesh size. For example, away from the inner boundary the second x derivative of $U_{i,j}$ was approximated by

$$\frac{\partial^2 U_{i,j}}{\partial x^2} \approx \frac{1}{h^2} (U_{i-1,j} - 2U_{i,j} + U_{i+1,j}),$$

but near the inner boundary, where, say, node $(i-1, j)$ is at distance δh from node (i, j), with $0 < \delta \leqslant 1$, the approximation was modified to

$$\frac{\partial^2 U_{i,j}}{\partial x^2} \approx \frac{2}{(\delta + \delta^2)\, h^2} (U_{i-1,j} - (1+\delta)\, U_{i,j} + \delta U_{i+1,j}).$$

Analogous modifications were made for points where the mesh size to the right of the point of discretization was on the boundary and for y derivatives.

$F(x, y)$ and $G(x, y)$ were injected, and the resulting set of linear algebraic equations was solved iteratively until residuals were reduced below 10^{-4}.

Due to the smoothness of the solution, even relatively coarse grids provided excellent approximations to the fine grid solution. Also, there was very little dependence on the magnitude of ε, so long as it was not large compared to h. However, the upwind solution differed quite considerably from the solution obtained with isotropic diffusivity regardless of the mesh size, as predicted.

In Fig. 2 the solutions along the x axis from the outer boundary to the inner boundary are compared. The results depicted are virtually indistinguishable for any $h \leqslant \frac{1}{32}$ and $\varepsilon \leqslant 0.1h$ and clearly show that the upwind solutions are totally inadequate.

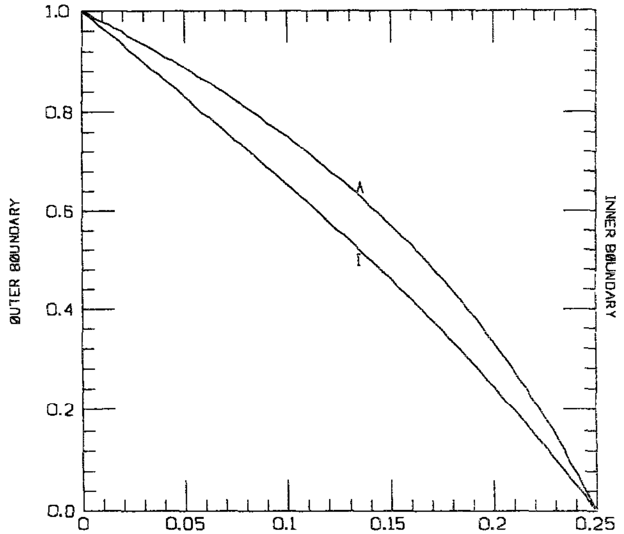

FIG. 2. Solution of (4.1) along centerline: A, upwind differencing; I, Isotropic artificial viscosity.

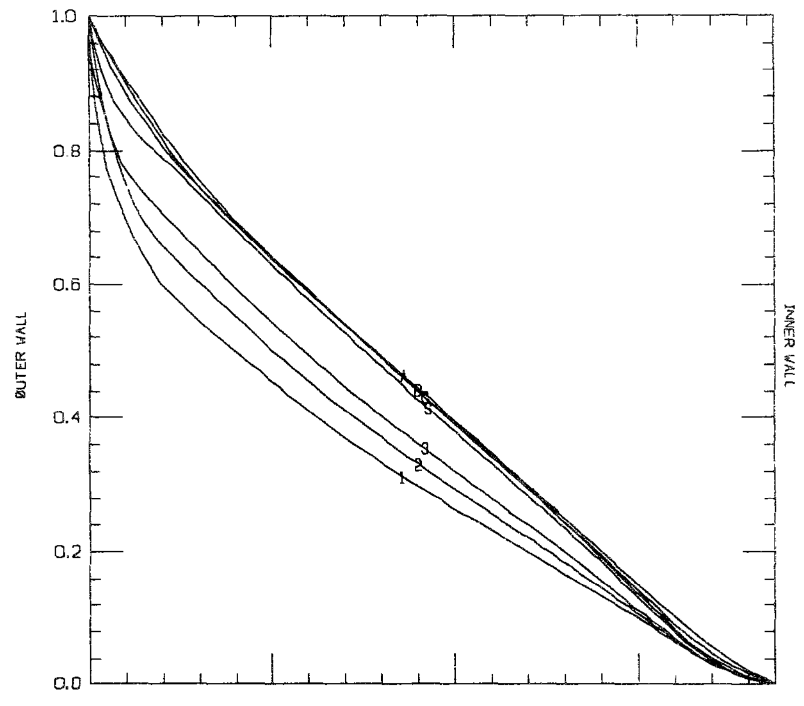

FIG. 3. Velocity along centerline. Comparison of numerical solutions of the incompressible Navier–Stokes equations by two different discretizations on progressively finer grids, from 32 by 32 for solutions *A* and 1 to 128 by 128 for solutions C and 3. Solutions A, B, and C were obtained with isotropic artificial viscosity, and solutions 1, 2, 3, and S were obtained with upwind differencing. S solved on a grid of 256 by 256 and corrected to second-order accuracy by defect corrections.

4.2. *Incompressible Navier–Stokes System*

The code described in [3], which had produced results that compared well with several published solutions of the driven cavity problem at high Reynolds numbers, was employed to depict the inadequacy of the upwind difference scheme in recirculating flow. In order to separate between the different aspects that are present in the driven cavity problem, many of which are caused by boundary layers, and yet retain simplicity of the domain and alignment of grid lines with the boundaries, the following model problem was chosen: steady incompressible flow in a square cavity of side 1, with a square hole of side $\frac{1}{4}$ in its middle. The boundaries are all parallel to the cartesian grid lines. The tangential velocities at the outer walls are given by $u_t = \sin \pi\xi$, ξ varying from 0 to 1 along the wall, and u_t driving the flow in a clockwise direction. The normal velocities at the outer walls and all velocities at the inner walls are equal to zero.

The problem was solved on grids varying from 32 by 32 to 128 by 128 grid intervals with both an upwind difference scheme and isotropic artificial viscosity. A second-order solution was computed on a 256 by 256 grid using upwind differencing and cycles of defect corrections. It is clear from Fig. 3 that the upwind solutions are very unsatisfactory on all grids, particularly coarse ones.

5. Conclusion

The proofs and the numerical examples make it clear that anisotropic artificial viscosity may lead to erroneous results, even in the most basic of problems, where exact solutions can be obtained. The bad approximation often goes unnoticed for two main reasons. One is that most interesting problems have solutions with boundary layers, the resolution of which requires small cross-sream viscosity, which is easily obtainable by upwind differencing when streamlines are aligned with the grid. In such cases the upwind difference results may be considerably better than those obtained with isotropic viscosity on the same grid. The other reason is that the error caused by anisotropic artificial viscosity is strongly dependent on the curvature of the solution. In fact, in some other common cases the anisotropic viscosity scheme can be shown to converge to the correct solution.

We make no claim that upwind difference schemes should not be used in recirculating flow problems. However, since the first-order upwind scheme does not, in the general case, yield first-order accurate solutions, it is doubtful whether it is the most efficient tool to be used for reaching the ultimate goal of second-order solutions to general incompressible flow problems in just a few minimal work units. Higher order upwind schemes may of course yield better solutions, but there are many well-known problems associated with high-order schemes. Moreover, it is still important that the cross-stream behavior be determined by physical-like viscosity in certain recirculating flow problems.

6. A Note on Multigrid Research

The trouble reported herein was first detected while attempting to develop fast multigrid solvers for high-Reynolds incompressible flows with separation and flows in closed vessels. Slow multigrid convergence for such flows has bothered several researchers (usually without their being conscious of the underlying reason: most investigators are generally all too easily resigned to having very graded multigrid performances.) Upon examination we traced the problem to the poor approximation obtained for such flows by upwind differencing. The bad discretization breeds poor multigrid convergence (since coarse grids do not yield proper approximations to smooth fine-grid errors), but, more important, it is a trouble by itself, which was not sufficiently recognized before.

This, in fact, is another good reason for using multigrid solvers: they force one to use good discretization schemes. A bad discretization will not pass unnoticed, since it is detected by the multigrid convergence rates. The implications of the findings reported herein for the design of multigrid solvers will be described elsewhere.

Appendix A

Choosing c large enough, say 10, three integrations of the terms in (3.9) yield

$$\int_a^R \int_a^t \int_a^s (r\varphi'(r))'\, dr\, ds\, dt$$

$$= -\frac{(R-a)^2}{2} a\varphi'(a) + \int_a^R \left[t\varphi(t) - \int_a^t \varphi(s)\, ds \right] dt \tag{A1}$$

with

$$\left| \int_a^R \left[t\varphi(t) - \int_a^t \varphi(s)\, ds \right] dt \right|$$

$$\leqslant \int_a^R t\,|\varphi(t)|\, dt + \int_a^R \frac{1}{a} \int_a^t s\,|\varphi(s)|\, ds \leqslant \frac{2Rc}{a} \cdot \|u_\theta\|$$

$$\left| \int_a^R \int_a^t \int_a^s [\psi'(r) - \varphi'(r)]\, dr\, ds\, dt \right|$$

$$\leqslant \int_a^R \int_a^t [|\psi(s)| + |\varphi(s)|]\, ds\, dt \leqslant \frac{2Rc}{a} \cdot \|u_\theta\| \tag{A2}$$

$$\left|\int_a^R \int_a^t \int_a^s \frac{1}{r}\left[-\psi(r)+\eta(r)-2\kappa(r)\right] dr\, ds\, dt\right| \leqslant \frac{2R^2 c}{a^2}\cdot(\|u_\theta\|+\|u_r\|) \tag{A3}$$

$$\begin{aligned}\int_a^R \int_a^t \int_a^s &\left[r(r\gamma'(r))'-\gamma(r)\right] dr\, ds\, dt\\ &=\int_a^R \int_a^t \left[s^2\gamma'(s)-a^2\gamma'(a)-s\gamma(s)\right] ds\, dt\\ &=-\frac{(R-a)^2}{2}a^2\gamma'(a)+\int_a^R t^2\gamma(t)\, dt-\int_a^R\int_a^t 3s\gamma(s)\, ds\, dt\end{aligned} \tag{A4}$$

with

$$\left|\int_a^R t^2\gamma(t)\, dt-\int_a^R\int_a^t 3s\gamma(s)\, ds\, dt\right| \leqslant 4Rc\cdot\|u_\theta\|$$

$$\begin{aligned}\int_a^R \int_a^t \int_a^s \int_0^{2\pi} &\left[r^2 u_r(r,\theta)\frac{\partial u_\theta(r,\theta)}{\partial r}+r u_r(r,\theta)\cdot u_\theta(r,\theta)\right] d\theta\, dr\, ds\, dt\\ &=\int_a^R\int_a^t\int_0^{2\pi}\left\{s^2 u_r(s,\theta)\cdot u_\theta(s,\theta)\right.\\ &\quad\left.-\int_a^s\left[r u_r(r,\theta)+r^2\frac{\partial u_r(r,\theta)}{\partial r}\right]u_\theta\, dr\right\} d\theta\, ds\, dt\\ &=\int_a^R\int_a^t\int_0^{2\pi} s^2 u_r(s,\theta)\cdot u_\theta(s,\theta)\, d\theta\, ds\, dt\\ &\quad+\int_a^R\int_a^t\int_a^s\int_0^{2\pi}\frac{r}{2}\frac{\partial(u_\theta(r,\theta))^2}{\partial\theta}\, d\theta\, dr\, ds\, dt.\end{aligned} \tag{A5}$$

The last term, obtained by substitution from (3.4c) and (3.6), vanishes in the integration with respect to θ, and

$$\begin{aligned}&\left|\int_a^R\int_a^t\int_0^{2\pi} s^2 u_r(s,\theta)\cdot u_\theta(s,\theta)\, d\theta\, ds\, dt\right|\\ &\qquad\leqslant\int_a^R t\,\|u_r u_\theta\|\, dt\leqslant\frac{R^2}{2}\|u_r u_\theta\|\leqslant\frac{R^2}{2}\|u_r\|\cdot\|u_\theta\|\end{aligned}$$

and, finally,

$$\int_a^R\int_a^t\int_a^s\frac{1}{r^2}\, dr\, ds\, dt=\frac{R^2-a^2}{2a}-R\ln\frac{R}{a}. \tag{A6}$$

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